Elliptic Function Solutions of (2+1)-dimensional Long Wave – Short Wave Resonance Interaction Equation via a sinh-Gordon Expansion Method

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With the aid of symbolic computation, the sinh-Gordon equation expansion method is extended to seek Jacobi elliptic function solutions of (2+1)-dimensional long wave-short wave resonance interaction equation, which describe the long and short waves propagation at an angle to each other in a two-layer fluid. As a result, new Jacobi elliptic function solutions are obtained. When the modulus *m* of Jacobi elliptic functions approaches 1, we also deduce the singular oliton solutions; while when the modulus $m \to 0$, we get the trigonometric function solutions. — PACS: 02.30.Jr, 03.40.Kf

Key words: Nonlinear Wave Equation; sinh-Gordon Equation; Jacobi Elliptic Function; Soliton Solution.

There exist many nonlinear wave equations in fluid mechanics, such as the Korteweg-de Vries equation, the Boussinesq equation, the (2+1)-dimensional dispersive long wave equation in shallow water, etc. [1]. Equations which describe the interaction of long and short waves have also been considered [2 – 4]. A simple scenario is the propagation of waves in a two-layer fluid, where the long and short waves are the interfacial and the surface waves, respectively. In 1989, Oikawa et al. [5] extended such long-short resonance to (2+1)-dimensions by incorporating the propagation of oblique waves. The (2+1)-dimensional long wave-short wave resonance interaction equation can be written as

$$S_{xx} - LS - i(S_t + S_y) = 0, \tag{1}$$

$$L_t - (2SS^*)_x = 0, \tag{2}$$

where L and S denote the long interfacial wave and the short surface wave packets, respectively, and S^* is the complex conjugate of S. This system describes the long and short wave propagation at an angle to each other in a two-layer fluid. It has been shown that (1) and (2) possess the bright and the dark double-soliton solutions [6], position and dromion solutions [7], and coherent soliton structures [8]. To our knowledge, the elliptic function solutions of this system were not considered before. More recently we presented a powerful method [9] to seek Jacobi elliptic function solutions based on the sinh-Gordon equation. In this paper we will extend this method to (1) and (2). With the aid of symbolic computation, we obtain three families of Jacobi elliptic function solutions. When the modulus of Jacobi elliptic functions approaches 1, we also deduce the singular soliton solutions while when the modulus is closest to 0, we get the trigonometric function solutions.

In order to solve (1) and (2) by using our method, we first reduce (1) and (2) to a system of ordinary differential equations. We make the transformations

$$S(x,y,t) = S(\xi)\exp(i\eta), \quad L(x,y,t) = L(\xi), \tag{3}$$

$$\xi = k(x + ly - \lambda t), \quad \eta = \alpha x + \beta y + \gamma t, \tag{4}$$

where ξ and η are real parameters and $k, l, \lambda, \alpha, \beta, \gamma$ are constants.

The substitution of (3) and (4) into (1) and (2) yields

$$i(2\alpha - \lambda - l)k\exp(i\eta)\frac{\mathrm{d}S}{\mathrm{d}\xi} + \left[k^2\frac{\mathrm{d}^2S}{\mathrm{d}\xi^2} + (\gamma+\beta)S - SL\right]\exp(i\eta) = 0, \tag{5}$$

$$\lambda\frac{\mathrm{d}L}{\mathrm{d}\xi} - 2\frac{\mathrm{d}S^2}{\mathrm{d}\xi} = 0. \tag{6}$$

From (6) we get

$$L = \frac{1}{\lambda}(2S^2 + C), \tag{7}$$

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where C is the constant of integration. From (5) we have

$$2\alpha-\lambda-l=0, \quad i.e., \quad \lambda=2\alpha-l, \tag{8}$$

and

$$k^2\frac{d^2S}{d\xi^2}+(\gamma+\beta)S-SL=0, \tag{9}$$

From (7) and (9) we eliminate L and get

$$k^2\frac{d^2S}{d\xi^2}+(\gamma+\beta-\frac{C}{\lambda})S-\frac{2}{\lambda}S^3=0. \tag{10}$$

Therefore we use (3), (4), (7), (8) to reduce (1) and (2) to one nonlinear ODE (10). If we can find elliptic function solutions of (10), we can obtain the corresponding solutions of (1) and (2).

According to the sinh-Gordon expansion method [9] we assume that (10) has the solution

$$S(\xi)=A_0+A_1\sinh w(\xi)+B_1\cosh w(\xi), \tag{11}$$

where the new variable w satisfies the equation

$$\frac{d^2w}{d\xi^2}=\sinh w\cosh w, \tag{12}$$

or in another form

$$(\frac{dw}{d\xi})^2=\sinh^2 w+1-m^2, \tag{13}$$

Remark: It is easy to see that (12) or (13) has the solution

$$\sinh[w(\xi)]=\mathrm{cs}(\xi;m) \tag{14a}$$

or

$$\cosh[w(\xi)]=\mathrm{ns}(\xi;m), \tag{14b}$$

where $\mathrm{cs}(\xi;m)$ and $\mathrm{ns}(\xi;m)$ are Jacobi elliptic functions and have the properties [10]

$$\begin{aligned}\frac{d\mathrm{cs}(\xi;m)}{d\xi}&=-\mathrm{ns}(\xi;m)\mathrm{ds}(\xi;m),\\ \frac{d\mathrm{ns}(\xi;m)}{d\xi}&=-\mathrm{cs}(\xi;m)\mathrm{ds}(\xi;m),\\ \mathrm{ns}^2(\xi;m)&=1+\mathrm{cs}^2(\xi;m).\end{aligned} \tag{15}$$

Substituting (11) into (10) along with (12) and (13), with the aid of symbolic computation we get the system of equations

$$-6A_1^2B_1-2B_1^3+2B_1k^2\lambda=0, \tag{16}$$

$$-CA_0+\gamma\lambda A_0-2A_0^3+6A_0A_1^2+\lambda\beta A_0=0, \tag{17}$$

$$\begin{aligned}&-6A_0^2B_1+(-C+\lambda\gamma+\lambda\beta)B_1+6B_1A_1^2\\&\quad-2\lambda k^2B_1+k^2\lambda(1-m^2)B_1=0.\end{aligned} \tag{18}$$

$$\begin{aligned}&(\beta\lambda+\gamma\lambda-C-k^2\lambda)A_1+k^2\lambda(1-m^2)A_1\\&\quad-6A_0^2A_1+2A_1^3=0,\end{aligned} \tag{19}$$

$$-12A_0A_1B_1=0, \tag{20}$$

$$-6A_0A_1^2-6A_0B_1^2=0, \tag{21}$$

$$-6B_1^2A_1-2A_1^3+2k^2\lambda A_1=0. \tag{22}$$

With the aid of Maple we solve the system and get the following solutions:

$$\begin{aligned}&A_0=B_1=0,\\&k=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{\lambda(2-m^2)}},\ A_1=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2-m^2}},\end{aligned} \tag{23}$$

$$\begin{aligned}&A_0=A_1=0,\\&k=\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{\lambda(1+m^2)}},\ B_1=\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{1+m^2}},\end{aligned} \tag{24}$$

$$\begin{aligned}&A_0=0,\\&k=\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}},\\&A_1=B_1=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2(1-2m^2)}},\end{aligned} \tag{25}$$

$$\begin{aligned}&A_0=0,\\&k=\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}},\\&A_1=-B_1=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2(1-2m^2)}}.\end{aligned} \tag{26}$$

Therefore we get three Jacobi elliptic function solutions:

$$\begin{aligned}S_1(x,y,t)&=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2-m^2}}\\&\cdot\ \mathrm{cs}\left[\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{\lambda(2-m^2)}}(x-ly+\lambda t)\right]\\&\cdot\ \exp[i(\alpha x+\beta y+\gamma t)],\end{aligned} \tag{27a}$$

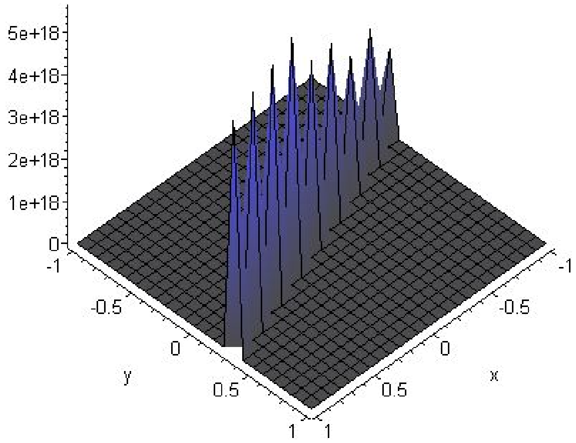

Fig. 1. L_1 in (27b) at $t = 0$ with $C = 0$, $m = 1$, $\lambda = -1$, $\beta = 1$, $\gamma = -2, l = 3$.

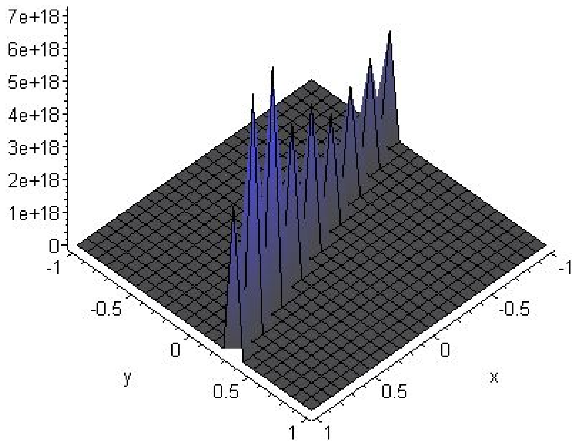

Fig. 2. L_1 in (27b) at $t = 0$ with $C = 0$, $m = 1/2$, $\lambda = -1$, $\beta = 1$, $\gamma = -2; l = 3$.

$$L_1(x,y,t) = \frac{2[C - \lambda(\gamma+\beta)]}{\lambda(2-m^2)} \cdot \mathrm{cs}^2\left[\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{\lambda(2-m^2)}}(x - ly + \lambda t)\right] + \frac{C}{\lambda}, \quad (27b)$$

$$S_2(x,y,t) = \pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{1+m^2}} \cdot \mathrm{ns}\left[\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{\lambda(1+m^2)}}(x - ly + \lambda t)\right] \cdot \exp[i(\alpha x + \beta y + \gamma t)], \quad (28a)$$

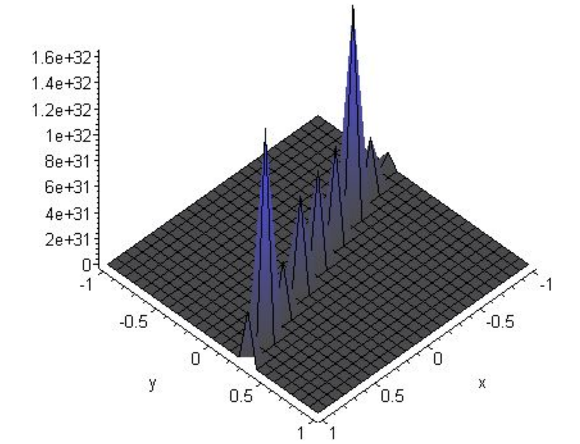

Fig. 3. L_1 in (27b) at $t = 0$ with $C = 0$, $m = 0$, $\lambda = -1$, $\beta = 1$, $\gamma = -2, l = 3$.

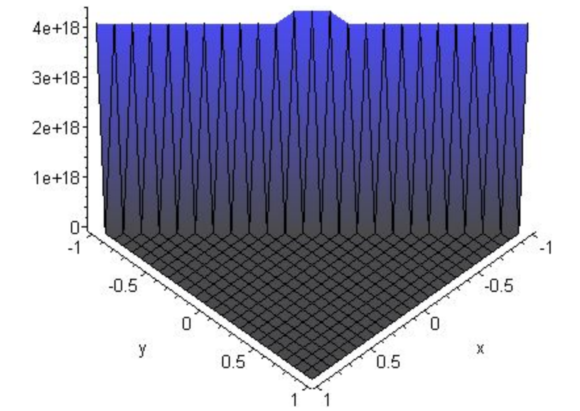

Fig. 4. L_2 in (28b) at $t = 0$ with $C = 0$, $m = 1$, $\lambda = 1$, $\beta = -1$, $\gamma = 2, l = -1$.

$$L_2(x,y,t) = \frac{2[\lambda(\gamma+\beta) - C]}{\lambda(1+m^2)} \cdot \mathrm{ns}^2\left[\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{\lambda(1+m^2)}}(x - ly + \lambda t)\right] + \frac{C}{\lambda}, \quad (28b)$$

$$S_3(x,y,t) = \pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2(1-2m^2)}} \cdot \left\{\mathrm{cs}\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}}(x - ly + \lambda t)\right] + \mu\,\mathrm{ns}\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}}(x - ly + \lambda t)\right]\right\} \cdot \exp[i(\alpha x + \beta y + \gamma t)], \quad (29a)$$

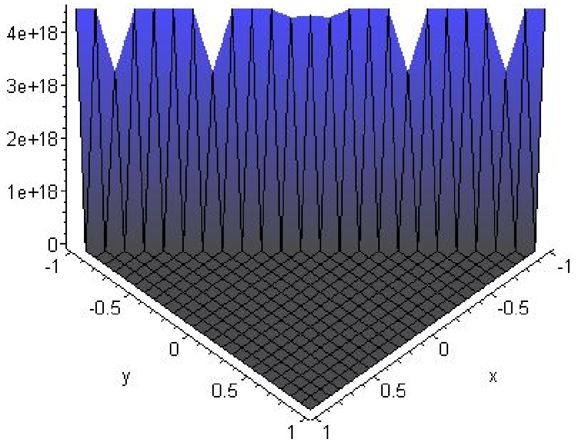

Fig. 5. L_2 in (28b) at $t = 0$ with $C = 0$, $m = 1/2$, $\lambda = 1$, $\beta = -1$, $\gamma = 2, l = -1$.

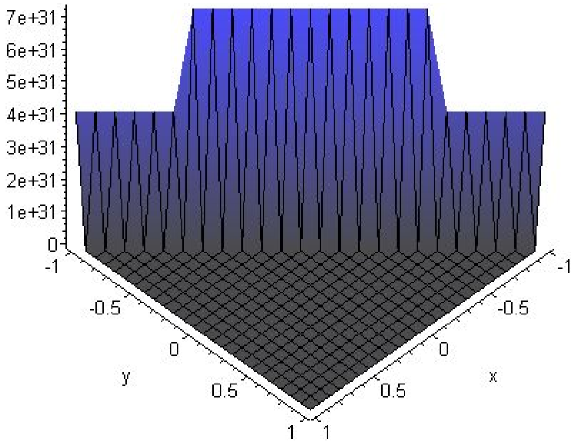

Fig. 6. L_2 in (28b) at $t = 0$ with $C = 0$, $m = 0$, $\lambda = 1$, $\beta = -1$, $\gamma = 2, l = -1$.

$$L_3(x,y,t) = \frac{C-\lambda(\gamma+\beta)}{\lambda(1-2m^2)} \cdot \left\{ 2\mathrm{cs}^2\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}}(x-ly+\lambda t)\right]+1 + 2\mu\,\mathrm{cs}\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}}(x-ly+\lambda t)\right] \cdot \mathrm{ns}\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda(1-2m^2)}}(x-ly+\lambda t)\right]\right\}, \quad (29b)$$

where $\mu = \pm 1$. In particular, when the modulus $m \to 1$, the solutions (27)–(29) approach the singular soliton solutions

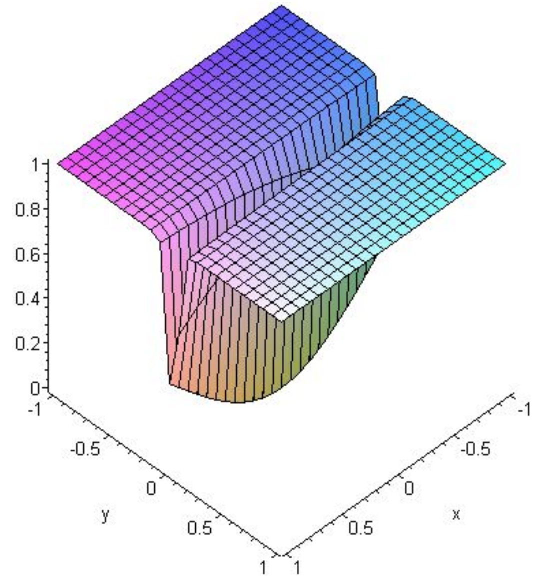

Fig. 7. L_3 in (29b) at $t = 0$ with $C = 0$, $m = 1$, $\lambda = 1$, $\beta = 1/2$, $\gamma = 1/2, l = 30$, $\mu = -1$.

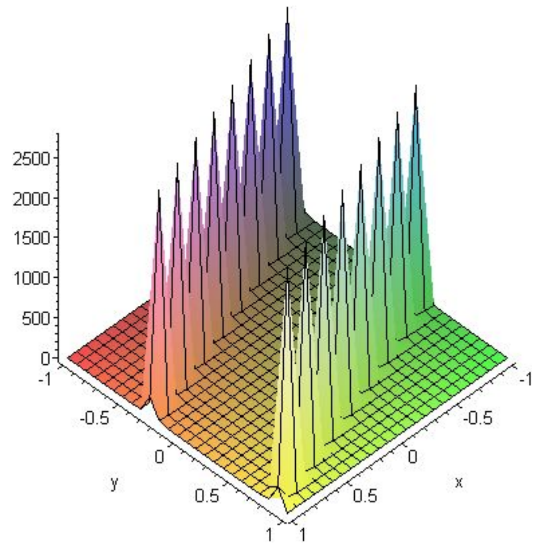

Fig. 8. L_3 in (29b) at $t = 0$ with $C = 0$, $m = 1/2$, $\lambda = 1$, $\beta = 1$, $\gamma = -2, l = 3, \mu = -1$.

$$S_4(x,y,t) = \pm\sqrt{C-\lambda(\gamma+\beta)} \cdot \mathrm{csch}\left[\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right] \cdot \exp\left[i(\alpha x+\beta y+\gamma t)\right], \quad (30a)$$

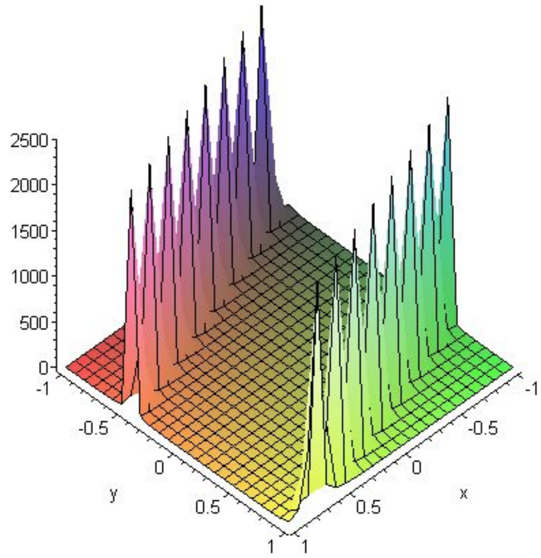

Fig. 9. L_3 in (29b) at $t=0$ with $C=0$, $m=0$, $\lambda=1$, $\beta=1$, $\gamma=-2$, $l=3$, $\mu=-1$.

$$L_4(x,y,t)=\frac{2[C-\lambda(\gamma+\beta)]}{\lambda}\cdot\operatorname{csch}^2\left[\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right]+\frac{C}{\lambda}, \quad (30b)$$

$$S_5(x,y,t)=\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{2}}\cdot\coth\left[\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{2\lambda}}(x-ly+\lambda t)\right]\cdot\exp[i(\alpha x+\beta y+\gamma t)], \quad (31a)$$

$$L_5(x,y,t)=\frac{\lambda(\gamma+\beta)-C}{\lambda}\cdot\coth^2\left[\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{2\lambda}}(x-ly+\lambda t)\right]+\frac{C}{\lambda}, \quad (31b)$$

$$S_6(x,y,t)=\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{2}}\cdot\left\{\operatorname{csch}\left[\pm\sqrt{\frac{2\lambda(\gamma+\beta)-2C}{\lambda}}(x-ly+\lambda t)\right]+\mu\coth\left[\pm\sqrt{\frac{2\lambda(\gamma+\beta)-2C}{\lambda}}(x-ly+\lambda t)\right]\right\}\cdot\exp[i(\alpha x+\beta y+\gamma t)], \quad (32a)$$

$$L_6(x,y,t)=\frac{\lambda(\gamma+\beta)-C}{\lambda}\cdot\left\{\{2\operatorname{csch}^2\left[\pm\sqrt{\frac{2\lambda(\gamma+\beta)-2C}{\lambda}}(x-ly+\lambda t)\right]+1+2\mu\operatorname{csch}\left[\pm\sqrt{\frac{2\lambda(\gamma+\beta)-2C}{\lambda}}(x-ly+\lambda t)\right]\cdot\coth\left[\pm\sqrt{\frac{2\lambda(\gamma+\beta)-2C}{\lambda}}(x-ly+\lambda t)\right] \quad (32b)$$

These singular solutions imply that at a certain time $t=t_0$, there exists the point (x_0,y_0) at which the wave will blow up.

While, when the modulus $m\to 1$, the solutions (27)–(29) approach the trigonometric function solutions

$$S_7(x,y,t)=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2}}\cdot\cot\left[\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2\lambda}}(x-ly+\lambda t)\right]\cdot\exp[i(\alpha x+\beta y+\gamma t)], \quad (33a)$$

$$L_1(x,y,t)=\frac{C-\lambda(\gamma+\beta)}{\lambda}\cdot\cot^2\left[\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2\lambda}}(x-ly+\lambda t)\right]+\frac{C}{\lambda}, \quad (33b)$$

$$S_8(x,y,t)=\pm\sqrt{\lambda(\gamma+\beta)-C}\cdot\csc\left[\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{\lambda}}(x-ly+\lambda t)\right]\cdot\exp[i(\alpha x+\beta y+\gamma t)], \quad (34a)$$

$$L_8(x,y,t)=\frac{2[\lambda(\gamma+\beta)-C]}{\lambda}\cdot\csc^2\left[\pm\sqrt{\frac{\lambda(\gamma+\beta)-C}{\lambda}}(x-ly+\lambda t)\right]+\frac{C}{\lambda}, \quad (34b)$$

$$S_9(x,y,t)=\pm\sqrt{\frac{C-\lambda(\gamma+\beta)}{2}}\cdot\left\{\cot\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right]\right.$$

$$+\,\mu\csc\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right]\Bigg\}$$

$$\cdot\ \exp[i(\alpha x+\beta y+\gamma t)], \tag{35a}$$

$$L_9(x,y,t)=\frac{C-\lambda(\gamma+\beta)}{\lambda}$$

$$\cdot\left\{2\cot^2\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right]+1\right.$$

$$+\,2\mu\cot\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right]$$

$$\left.\cdot\ \csc\left[\pm\sqrt{\frac{2C-2\lambda(\gamma+\beta)}{\lambda}}(x-ly+\lambda t)\right]\right\}. \tag{35b}$$

The profiles of the above-mentioned solutions of (2) are described in Figures 1 – 9.

Remark: In the solutions (27b), (28b) and (29b), we take the positive sign +. Similarly, we also make the corresponding figures of the modulus $|S_i|$ of solutions $S_i(i=1,2,3)$ in (27a), (28a) and (29a).

In summary, we have extended the sinh-Gordon equation expansion method [9] to a pair of (2+1)-dimensional long wave-short wave resonance interaction equations. As a result, three Jacobi elliptic function solutions are obtained. When the modulus of the Jacobi elliptic functions approaches 1, we also deduce the singular soliton solutions; while when the modulus $m\to 0$, we get the trigonometric function solutions. Further study is needed to see whether (1) and (2) have other elliptic function solutions.

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